Notes (working draft) for Math 6201-01 – Topology Spring 2026 Semester

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February 19, 2026

Start: 1/6/2026

1 Intro

1.1 First Day Admin Stuff

- Instructor & Student introductions
- Go over syllabus
  - key content: Office Hours; Description & Course Goals; Homework & Gradescope; Presentations; Exams;
- students encouraged to **ask many questions** and provide feedback/criticism
- main text for the semester is Algebraic Topology by Allen Hatcher
  - Students are expected to **read the text on their own**
  - we cannot possibly cover everything, and there are many important things or technical proofs we'll have to gloss over
  - view the lectures as a **guide** to the key ideas, rather than a comprehensive treatment.

1.2 Broad Goals

Main goal of the semester is to develop sophisticated **algebraic invariants** that can be used to glean information about, and distinguish, topological spaces. Specifically these are:

- **fundamental group** $\pi_1(X)$ and higher **homotopy groups** $\pi_n(X)$
- **homology groups** $H_n(X)$ and the **cohomology ring** $H^*(X)$.

1.3 *Categories and functors*

The language of category theory is helpful to organize the structure involved with our invariants.

Definition 1.1. A **category** $\mathcal{C}$ consists of:

- a collection $\mathrm{Ob}(\mathcal{C})$ of **objects**
- a set $\mathrm{Mor}(X, Y)$ of **morphisms/arrows** for every pair $X, Y \in \mathrm{Ob}(\mathcal{C})$, with distinguished **identity morphisms** $\mathbb{1}_X \in \mathrm{Mor}(X, X)$
- **composition functions** $\circ\colon \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ satisfying

$$f \circ \mathbb{1} = f = 1 \circ f, \qquad \text{and} \quad (f \circ g) \circ h = f \circ (g \circ h)$$

Example 1.2.

- **Set** – category of sets with arbitrary functions as morphisms
- **Top** – category of topological spaces with continuous maps
- **Grp** – category of groups with homomorphisms (or subcagetory of abelian groups)
- $\mathbf{Mod}_R$ – category of modules over a fixed ring R with ring homomorphisms
- any group G can be viewed as a category with one object and G as the morphisms (with composition the group operation)

Definition 1.3. A (covariant) **functor** F from one category $\mathcal{C}$ to another $\mathcal{D}$ is as assignment sending

- each object $X \in \mathrm{Ob}(\mathcal{C})$ to an object $F(C) \in \mathrm{Ob}(\mathcal{D})$,
- each morphism $f \in \mathrm{Mor}(X, Y)$ to a morphism $F(f) \in \mathrm{Mor}(F(X), F(Y))$ so that $F(\mathbb{1}) = \mathbb{1}$ and $F(f \circ g) = F(f) \circ F(g)$.

There are similarly **contravariant** functors that reverse the direction of arrows.

Example 1.4.

- The "forgetful" functor from **Top** $\to$ **Set** sending a topological space to its underlying set.
- The "free" functor from **Set** $\to$ **Grp**, sending a set S to the free group on that set.
- The contravariant "vector space dual" functor from $\mathbf{Vect}_{\mathsf{F}}$ (category of vector spaces over a field F) to itself, sending a vector space V to its dual $V^* = \mathrm{Mor}(V, \mathsf{F}) = \mathrm{Hom}(V, \mathsf{F})$

recall that the set $\mathrm{Hom}(V, W)$ of linear transformations between two vector spaces is itself a vector space

1.4 *Homotopy*

Definition 1.5. Two maps $f, g \colon X \to Y$ are **homotopic**, denoted $f \simeq g$, if there is a family of maps $h_t \colon X \to Y$ for $t \in I = [0,1]$ so that:

- $f = h_0$ and $g = h_1$,
- the map $H \colon X \times I \to Y$ given by $H(x,t) = h_t(x)$ is continuous.

> note this implies each $h_t \colon X \to Y$ is continuous

The family h_t is called a **homotopy** between $h_0 = f$ and $h_1 = g$.
A map is **nullhomotopic** if it is homotopic to a constant map.

Definition 1.6. A map $f \colon X \to Y$ is a **homotopy equivalence** if there is a map $g \colon Y \to X$ so that $gf \simeq \mathbb{1}_X$ and $fg \simeq \mathbb{1}_Y$.

In this case we say X and Y are **homotopy equivalent** or have the same **homotopy type**.

Fact 1.7. *Homotopy type is an equivalence relation!*

> Why? It is clearly symmetric and reflexive ($\mathbb{1}_X$ is a homotopy equivalence from X to itself), and transitivity holds since the composition of homotopy equivalences is a homotopy equivalence.

Example 1.8. Euclidean space $\mathbb{R}^n$ is homotopy equivalent to the one-point space $Y = \{*\}$. Indeed, let $g \colon Y \to \mathbb{R}^n$ be the inclusion of the origin and $f \colon \mathbb{R}^n \to Y$ the unique map. Then $fg = \mathbb{1}_Y$, and $gf \colon \mathbb{R}^n \to R^n$ is the constant map sending everything to the origin. Now define

$$H \colon \mathbb{R}^n \times I \to \mathbb{R}^n \quad \text{by} \quad H(x,t) = f_t(x) = tx.$$

Clearly H is continuous and thus gives a homotopy between $h_1 = \mathbb{1}_{\mathbb{R}^n}$ and $h_0 = gf$ the constant map. Thus f and g are homotopy equivalences between $\mathbb{R}^n$ and Y.

Definition 1.9. A space X is **contractible** if it has the homotopy type of a point.

Exercise 1.10. X contractible $\iff$ identity $\mathbb{1}_X$ is nullhomotopic

> as is clear from the example

We shall also need a variation of homotopy:

Definition 1.11. A homotopy $h_t \colon X \to Y$ such that the restrictions $h_t|_A$ to some subspace $A \subset X$ are independent of t is called a **homotopy relative to** A or a **homotopy rel** A.

> meaning $h_t|_A = h_s|_A$ for all $t, s \in I$

Example 1.12. A **deformation retraction** of X onto a subspace $A \subset X$ is a homotopy $f_t \colon X \to X$ so that $f_0 = \mathbb{1}_X$, $f_t|_A = \mathbb{1}_A$ and $f_1(X) = A$.

> Note that normal homotopy is the same thing as a homotopy rel the emptyset $\emptyset$

- In this case $f_1 \colon X \to A$ is a homotopy equivalence with homotopy the inclusion $A \hookrightarrow X$; thus X and A have the same homotopy type!

Proposition 1.13. *Homotopy rel a subspace $A \subset X$ is an equivalence relation on the set of maps $X \to Y$. The equivalence class of $f\colon X \to Y$ is called its* ***homotopy class*** *and denoted $[f]$.*

Proof.

- reflexivity $f \simeq f$ is witnessed by the constant homotopy $f_t = f$.
- (symmetry) if f_t is a homotopy rel A from f_0 to f_1, then the inverse homotopy $\bar{f}_t = f_{1-t}$ gives a homotopy rel A from f_1 to f_0.
- (transitivity) say f_t is a homotopy rel A from f_0 to f_1 and g_t a homotopy rel A from $f_1 = g_0$ to g_1. Define

$$h_t(x) = \begin{cases} f_{2t}(x), & 0 \le t \le 1/2 \\ g_{2t-1}(x), & 1/2 \le t \le 1 \end{cases}$$

  Then the restrictions $h_t|_A$ are independent of t, and restrictions of $H(t,x) = h_t(x)$ to $X \times [0, 1/2]$ and $X \times [1/2, 1]$ are agree on the overlap $X \times \{1/2\}$. Thus H is continuous by the gluing lemma and we have a homotopy from $h_0 = f_0$ to $h_1 = g_1$.

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1.5 Cell complexes

Before going on, it is nice to discuss more constructions to build interesting examples.

First, recall **attaching spaces**: given $A \subset X$ closed and $f\colon A \to Y$ continuous, we build

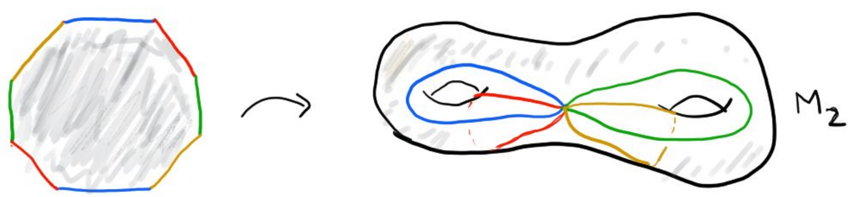
$$Y \sqcup_f X = Y \sqcup X/(a \sim f(a)) \quad \text{for } a \in A$$

Definition 1.14. A **cell complex** or **CW** complex is a space X constructed as follows:

- Start with discrete space X^0, whose points regarded as 0–cells.
- Inductively, form n–skeleton X^n from X^{n-1} by attaching n–cells e^n_α via maps $\varphi_\alpha\colon S^{n-1} \to X^{n-1}$. Thus: X^n is the the attaching space $X^{n-1} \bigsqcup_\alpha D^n_\alpha$ formed by attaching n–disks D^n_α to X^{n-1} via the maps.
  - As a set $X^n = X^{n-1} \cup_\alpha e^n_\alpha$, where each e^n_α is an open n–disk.
  - each cell has a **characteristic map** $\Phi_\alpha\colon D^n_\alpha \to X$ that extends the attaching map and gives a homeomorphism between the interior of D^n_α and e^n_α
- Either stop at a finite stage $X = X^n$, or continue forever yielding $X = \bigcup_n X^n$. In latter case, X given **weak topology**: wherein $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Example 1.15. Many familiar spaces have a natural cell structure:

- **Surface**: Torus T^2 built by identifying top and bottom of a square. More generally, the closed genus–g surface M_g build by identifying opposite sides of a $4g$–gon. In the quotient:
  - vertices of polygon $\rightsquigarrow$ a single 0-cell
  - edges of the polygon $\rightsquigarrow$ wedge of $2g$ circles (the 1 skeleton)
  - interior of the polygon $\rightsquigarrow$ single 2–cell



- A 1–dimensional CW complex is a **graph**: consists of vertices to which edges are attached.

Loops and multiple edges between vertices are allowed

- The sphere $\mathbb{S}^n$ has a cell structure with a single 0–cell e^0 and a single n–cell e^n attached by the constant map $\mathbb{S}^{n-1} \to e^0$.
- Real projective space $\mathbb{RP}^n$ is quotient of $\mathbb{S}^n$ by identifying antipodal points, equivalently D^n with antipodal points of $\partial D^n = \mathbb{S}^{n-1}$. That is $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching a single n–cell with quotient $\mathbb{S}^{n-1} \to \mathbb{RP}^{n-1}$. Inductively $\mathbb{RP}^n$ has a cell structure $e^0 \cup \cdots \cup e^n$ with one cell in each dimension $0, \ldots, n$.
- Similarly **complex projective space** $\mathbb{CP}^n$ is the set of complex lines (ie. 1-dimensional subspaces) in $\mathbb{C}^{n+1}$. Alternately, is the quotient of the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ under relation $v \sim \lambda v$ for $|\lambda| = 1$.
  - Each vector of $\mathbb{S}^{2n+1}$ is equivalent to a point in the set D^{2n}_+ of vectors with last coordinate nonnegative real, that is, of the form $\left(w, \sqrt{1-|w|^2}\right) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$. Thus D^{2n}_+ is a $2n$–disk!

    Namely, D^{2n}_+ is just the graph of $w \mapsto \sqrt{1-|w|^2}$.
  - Interior points of D^{2n}_+ are not identified with anything, and on its boundary $S^{2n-1} \subset S^{2n+1}$ we just have the identifications $v \sim \lambda v$ of $\mathbb{CP}^{n-1}$.
  - Thus $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a single $2n$–cell e^{2n} via the quotient map $\mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$.
  - Inductively, $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ gets a cell structure with one cell in each even dimension.

- We also have the **infinite-dimensional projective spaces**

$$\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n \quad \text{and} \quad \mathbb{CP}^\infty = \bigcup_n \mathbb{CP}^n$$

  with one cell in each dimension (resp. even dimension).

Definition 1.16. A **subcomplex** of a CW complex X is a closed subspace $A \subset X$ that is a union of cells. We call (X, A) a **CW pair**.

For example, each skeleton X^n of a cell complex is a subcomplex

- A is itself a CW complex, since being closed forces the characteristic map of each cell in A to have image in A

1.6 Operations on spaces

Many operations on CW complexes produce new CW complexes:

Products: If X and Y are CW complexes, then $X \times Y$ has a cell structure, with cells the products $e^n_\alpha \times e^m_\beta$ of the cells e^n_α of X and e^m_β of Y.

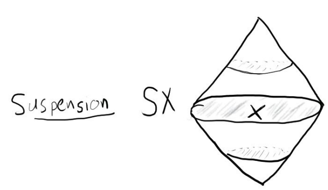
Caution: If X and Y have infinitely many cells, the CW-topology is sometimes finer than the product topology

Quotients: If (X, A) is a CW pair, then the quotient X/A inherits a cell structure:

- get one new 0–cell, the image of A in X/A
- for each cell e^n_α NOT in A we get a cell of X/A with attaching map the composition $S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/A^{n-1}$.

Example 1.17. Consider the genus-g surface $X = M_g$ with subcomplex $A = X^1$ the 1–skeleton. Then X/A is S^2, since it has exactly one 2–cell attached to one 0–cell (by the constant map)

Suspension: The **suspension** of X is the quotient SX of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point.



- if X is a CW complex, then SX inherits a cell structure (as iterated quotient of $X \times I$)
- Suspension is actually a functor: a map $f\colon X \to Y$ suspends to $Sf\colon SX \to SY$, the quotient of $f \times \mathbb{1}\colon X \times I \to Y \times I$.

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Example 1.18. If $X = \mathbb{S}^n$, then $SX = \mathbb{S}^{n+1}$

Wedge sum: Recall that the wedge $\bigvee_\alpha X_\alpha$ is the quotient of $\coprod_\alpha X_\alpha$ by identifying points $x_\alpha \in X_\alpha$ to a single point. If each X_α is a cell complex and the points x_α are 0–cells, then $\bigvee_\alpha X_\alpha$ is a cell complex (as the quotient by collapsing a subcomplex).

Example 1.19. For any cell complex X, the quotient X^n/X^{n-1} is a wedge sum $\bigvee_\alpha \mathbb{S}^n_\alpha$ of n–spheres (one for each n-cell e^n_α of X).

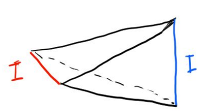
Join: The **join** of two spaces X and Y is the union $X * Y$ of all line segments joining points of X to points of Y. This is a quotient:

$$X * Y = X \times Y \times I/\sim \quad \text{where} \quad \begin{matrix} (x, y_1, 0) \sim (x, y_2, 0) \\ (x_1, y, 1) \sim (x_2, y, 1) \end{matrix}$$

collapsing $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y.

- Contains copies of X and Y at the two ends; every other point (x, y, t) lies on a line segment joining points x, y in these subspaces.
- Convenient to write points of as a formal sums $t_1x + t_2y$, where $t_i \geq 0$ and $t_1 + t_2 = 1$, subject to rules $0x + 1y = y$ and $1x + 0y = x$.
- Iterated n-fold join $X_1 * \cdots * X_n$ is space of formal linear combinations $t_1x_1 + \cdots + t_nx_n$, where $0 \leq t_i \leq 1$ and $t_1 + \cdots + t_n = 1$, with convention that $0x_i$ terms can be omitted.

Example 1.20. • The join $I * I$ of two intervals is a tetrahedron



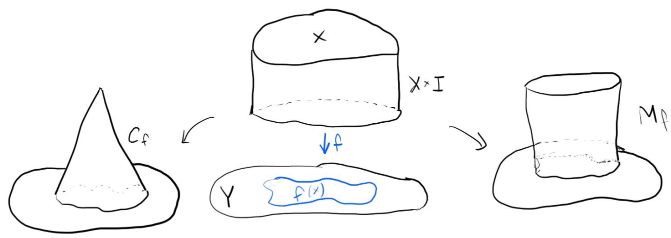
- The join of two points is the interval I.
- More generally join of $n+1$ points is the n **simplex** Δ^n. By taking the points to be the standard basis vectors of $\mathbb{R}^{n+1}$, this is concretely realized as:

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^n \mid t_0 + \cdots + t_n = 1 \text{ and } t_i \geq 0\}$$

Mapping cylinder/cone: Given a map $f : X \to Y$, its **mapping cylinder** M_f is the quotient of $(X \times I) \sqcup Y$ identifying each $(x, 1)$ with $f(x)$. The **mapping cone** C_f is the quotient of M_f that collapses $X \times \{0\}$ to a point:

> M_f can viewed as the attaching space, where $X \times \{1\} \subset X \times I$ is attached to Y via $f \circ \pi_X : X \times \{1\} \to Y$

> C_f can also be viewed as an attaching space $Y \sqcup_f CX$, where we attach the cone CX along $X \times \{1\}$ via the identifications $(x, 1) \sim f(x)$



- The mapping cylinder M_f always deformation retracts onto the subspace Y! (the homotopy h_s sends (x, t) to $(x, (1-s)t + s)$)
- CX = the **cone on** X = the collapse $X \times I/(X \times \{0\})$. Alternately:
  - CX is the mapping cone of the identity $\mathbb{1}_X \colon X \to X$
  - CX is also the mapping cylinder of the constant map $X \to \{*\}$ to a one point space. In particular, CX is always contractible, since it deformation retracts onto a point (the image of $X \times \{0\}$)!

1.7 *Criteria for homotopy equivalence*

It's helpful to have some ways to modify a space to produce something homotopy equivalent. We will use the following:

Definition 1.21. A pair (X, A), where A is a subspace of X, has the **homotopy extension property** if given any map $f_0 \colon X \to Y$, then any homotopy $f_t \colon A \to Y$ of $f_0|_A$ extends to a homotopy $f_t \colon X \to Y$.

Fact 1.22. *If (X, A) is a CW pair, then:*

- *(X, A) has the homotopy extension property.*
- *$X \times I$ deformation retracts onto $X \times \{0\} \cup A \times I$.*

Proof. See textbook. □

We can now give two criteria for homotopy equivalence:

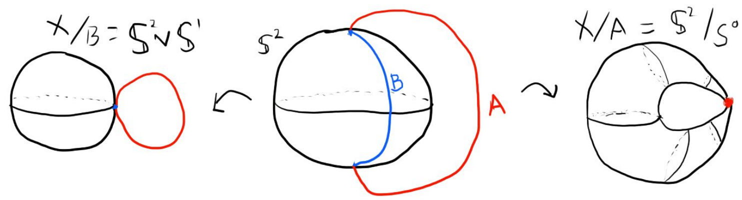
Proposition 1.23. *If (X, A) is a CW pair and A is contractible, then the quotient map $q \colon X \to X/A$ is a homotopy equivalence.*

Proof. Let $f_t \colon A \to A \subset X$ be a homotopy between $f_0 = \mathbb{1}_A$ and a constant map. May extend to a homotopy $f_t \colon X \to X$. The composition $qf_t \colon X \to X/A$ is constant on fibers and descends to $\overline{f}_t \colon X/A \to X/A$ satisfying $qf_t = q\overline{f}_t$. Now, $f_1 \colon X \to X$ sends A to a point and descends to $g \colon X/A \to X$ with $gq = f_1 \simeq f_0 = \mathbb{1}_X$.

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_t"'] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_1"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

It follows that $qg = \overline{f}_1 \simeq \overline{f}_0 = \mathbb{1}_{X/A}$. Thus q and g are homotopy inverses. $\square$

Example 1.24. Let X be the space obtained from $\mathbb{S}^2$ by attaching the ends of an arc A at the north and south poles. Also let B be an arc in $\mathbb{S}^2$ joining the poles. We can give X a CW structure so that A and B are both subcomplexes. As A and B are contractible, we see that $X \simeq X/A \simeq X/B$. Note that $X/A = \mathbb{S}^2/\mathbb{S}^0$ is the sphere with two points identified, and X/B is $\mathbb{S}^2 \vee \mathbb{S}^1$



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Proposition 1.25. *Let (X, A) be a CW pair. If two maps $f, g \colon A \to Y$ are homotopic, then the associated attaching spaces $Y \sqcup_f X$ and $Y \sqcup_g X$ are homotopy equivalent.*

Sketch. Let $F \colon A \times I \to Y$ be the homotopy $f \simeq g$. Then can use the deformation retractions $X \times I \to X \times \{i\} \cup A \times I$, for $i = 0, 1$, to get a deformation retraction of the attaching space $Y \sqcup_F (X \times I)$ onto both of the two subcomplexes $Y \sqcup_f X$ and $Y \sqcup_g X$. $\square$

Example 1.26. Again let $B \subset \mathbb{S}^2$ be a closed arc from the north to south pole. Let $f \colon B \to \mathbb{S}^1$ map B around $\mathbb{S}^1$. Since B is contractible, f is homotopic to the constant map $g \colon B \to \mathbb{S}^1$. Thus the two attaching spaces are homotopy equivalent. Note that $\mathbb{S}^1 \sqcup_f \mathbb{S}^2$ is the sphere $\mathbb{S}^2/\mathbb{S}^0$ with two points identified, and $\mathbb{S}^1 \sqcup_g \mathbb{S}^2$ is the wedge $\mathbb{S}^1 \vee \mathbb{S}^2$. So this is the same example as above.

Example 1.27. If (X, A) is a CW pair, then CA is a subcomplex of the mapping cone $C_\iota = X \cup CA$ of the inclusion $\iota \colon A \to X$.

- Thus $C_\iota \simeq C_\iota/CA = X/A$.
- Moreover, if $\iota \colon A \to X$ is nullhomotopic (ie "A is contractible in X) then $X/A \simeq C_\iota \simeq X \vee SA$, since C_ι and $X \vee SA$ are, respectively, the attaching spaces of CA to X along the ι and the constant map.

2 *The Fundamental Group*

Our first and simplest algebraic invariant is built by looking at homotopy classes of loops in a space.

Definition 2.1. A **path** in a space X is a continuous map $f\colon I \to X$.

- A **loop** is a path with the same "endpoints";
  we call $x_0 = f(0) = f(1)$ the **basepoint** of the loop.
- by a **homotopy of paths** we mean a homotopy f_t rel endpoints, that is, relative to the subspace $\{0,1\} \subset I$. Specifically, this means:
  - the map $F\colon I \times I \to X$ given by $F(s,t) = f_t(s)$ is continuous;
  - the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.

Since $I = \{*\} \times I$, one can view a path itself as a homotopy between two inclusions $\{*\} \to X$ of the one-point space! (namely the inclusions sending $*$ to $f(0)$ and $f(1)$)

Example 2.2 (Linear homotopy)**.** Any two paths f_0 and f_1 in $\mathbb{R}^n$ with the same endpoints x_0 and x_1 are homotopic via the linear homotopy

$$f_t(s) = (1-t)f_0(s) + tf_1(s)$$

During this homotopy each point $f_0(s)$ travels along the straight line segment from to $f_1(s)$.

More generally, if $C \subset \mathbb{R}^n$ is any convex subset, then any two maps $f_0, f_1\colon X \to C$ (that agree on a subset $A \subset X$) are homotopic (rel A) by a linear homotopy.

Definition 2.3. If $f, g\colon I \to X$ are two paths with $f(1) = g(0)$, we define the **concatenation** or **product path** $f \cdot g\colon I \to X$ by

$$(f \cdot g)(s) = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & 1/2 \le s \le 1 \end{cases}$$

viewing the paths f, g as homotopies, this is exactly the same concatenated homotopy we used above to show being homotopic is an equivalence relation

★ Concatenation respects homotopy! If f_t is a homotopy witnessing $f_0 \simeq f_1$ and g_t is a homotopy witnessing $g_0 \simeq g_1$ then $f_t \cdot g_t$ is a homotopy witnessing $f_0 \cdot g_0 \simeq f_1 \cdot g_1$. Thus we have a well-defined operation

$$[f] \cdot [g] = [f \cdot g] \qquad \text{for paths with } f(1) = g(0)$$

Notice that if we restrict our attention to loops based at a point $x_0 \in X$, then we can always concatenate since $f(1) = x_0 = g(0)$!

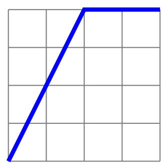
Definition 2.4. The **fundamental group** of X at basepoint x_0 is the set $\pi_1(X, x_0)$ of homotopy classes $[f]$ of loops $f\colon I \to X$ based at x_0.

Proposition 2.5. *$\pi_1(X, x_0)$ is a group with operation $[f][g] = [f \cdot g]$.*

Proof. We have a well-defined set with a well-defined binary operation. It remains to verify the group axioms.

To aid in this, a **reparametrization** of a path f is a composition $f\varphi$ where $\varphi\colon I \to I$ is any continuous map with $\varphi(0) = 0$ and $\varphi(1) = 0$. Note that $\varphi \simeq \mathbb{1}_I$ via a linear homotopy: $\varphi_t(s) = (1-t)\varphi(s) + ts$. Thus the composition $f\varphi_t$ gives a homotopy between $f\varphi$ and $f = f\mathbb{1}_I$. Thus: reparametrization preserves the homotopy class of a path!

Identity: If $f\colon I \to X$ is any path and c is the constant path at $f(1)$ (that is, $c(s) = f(1)$ for all $s \in I$), then $f \cdot c$ is the reparametrization of $f\varphi$ via the function φ with graph

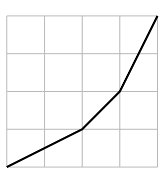


Thus $f \cdot c = f\varphi \simeq f$. Similarly, if c is the constant path at $f(0)$, then $c \cdot f \simeq f$. Therefore, turning our attention to loops, the identity element of $\pi_1(X, x_0)$ is the homotopy class of the constant path at x_0.

Inverses: The reverse of a path $f\colon I \to X$ is the path $\overline{f}\colon I \to X$ given by $\overline{f}(s) = f(1-s)$. Observe that the concatenation $f \cdot \overline{f}$ is homotopic to the constant path at $f(0)$ via the homotopy $h_t = f_t \cdot g_t$, where f_t agrees with f on $[0, 1-t]$ and is constant equal to $f(1-t)$ on $[1-t, t]$, and where $g_t = \overline{f_t}$. In particular, if f is a loop at x_0, then $[\overline{f}]$ is a two-sided inverse of $[f]$ in $\pi_1(X, x_0)$.

so f_t is a homotopy between f and the constant path at $f(0)$

Associativity: If f, g, h are paths with $f(1) = g(0)$ and $g(1) = h(0)$, we can form both concatenations $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$. These are different paths, but they are homotopic since one is a reparametrization of the other by a piecewise linear function φ with graph

reparam.  

homotopy

Thus the product in $\pi_1(X, x_0)$ is associative. □

Example 2.6. $\pi_1(\mathbb{R}^n, x_0) = 1$ is the trivial group. We have seen that any two maps $X \to \mathbb{R}^n$ are homotopic via a linear homotopy, so in particular, any loop $f\colon I \to X$ at x_0 is homotopic to the constant path at x_0.

Stop: 1/15/2026

Start: 1/20/2026

Dependence on basepoint

❓ How does the group $\pi_1(X, x_0)$ depend on the basepoint x_0?

- $\pi_1(X, x_0)$ only involves the path-component of X containing x_0
- for any point x_1 in same path component, can take a path h from x_0 to x_1.
- to each loop f at x_1 we can then associate a loop $h \cdot f \cdot \overline{h}$ at x_0.

Proposition 2.7. *The above gives a well-defined* ***change-of basepoint isomorphism****:*

$$\beta_h \colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad \textit{given by} \quad \beta_h[f] = [h \cdot f \cdot \overline{h}]$$

Proof. The map is well-defined, since if f_t is a homotopy of loops based at x_1 then $h \cdot f_t \cdot \overline{h}$ is a homotopy of loops based at x_0. The map is a homomorphism since $\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = \beta_h[f] \cdot \beta_h[g]$. Further, $\beta_{\overline{h}} \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ is clearly an inverse, hence β_h is an isomorphism. □

💬 As an abstract group $\pi_1(X, x_0)$ only depends on the path component of X containing x_0. Thus, when X is path-connected, we sometimes write $\pi_1(X)$ for shorthand, though technically one needs to specify a basepoint to view group elements as loops

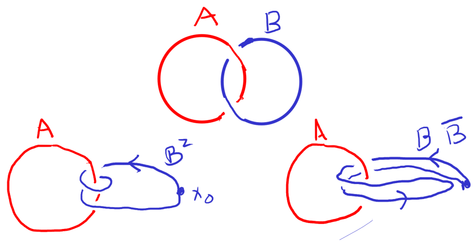
Definition 2.8. A space is **simply-connected** if it is path-connected and has trivial fundamental group. (This is equivalent to there being a unique homotopy class of paths connecting any two points.)

An intuitive motivating example

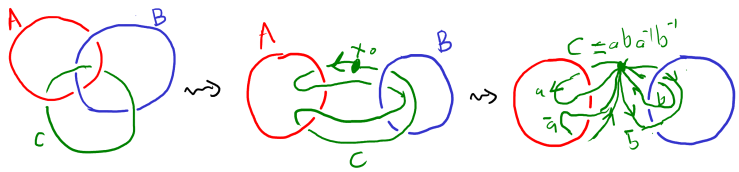
Before diving into technical details, lets look at an intuitive examples of linked vs unlinked loops in $\mathbb{R}^3$ to get a feel for the kinds of things the fundamental group might tell us:

Example 2.9. Linked loops in $\mathbb{R}^3$:

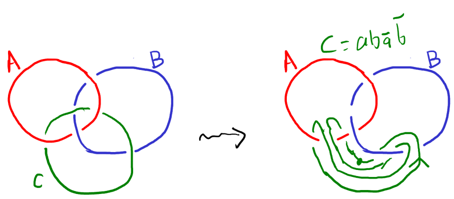
- View B as a loop in $X = \mathbb{R}^3 \setminus A$. If A and B are linked, intuitive that B should be nontrivial in $\pi_1(X)$. If A and B were unlinked, then B would be nullhomotopic in X; hence the fundamental group is a tool to verify that the loops are linked!



- Next consider Borromean rings A, B, C. The key feature is that if any loop is removed, the other two are not linked. Now view C as a loop in $X = \mathbb{R}^n \setminus (A \cup B)$ and realize as a product $ab\bar{a}\bar{b}$ of loops that each link once with A or B (in opposite directions). Fact that C is NOT nullhomotopic reflects that the commutator $aba^{-1}b^{-1}$ is nontrivial in $\pi_1(X)$. In fact, here $\pi_1(X)$ is the free group $F(a, b)$.



- Next adjust Borromean rings by swapping a crossing so that A and B are linked. Now see that $C = aba^{-1}b^{-1}$ can actually be unlinked from A and B! This shows the commutator is trivial in $\pi_1(X)$, which is in fact the free abelian group $\mathbb{Z}^2$ (here X is homotopy equivalent to the torus T^2).



2.1 *Covering spaces*

To move forward, we should actually calculate some examples. Our first goal is to calculate the fundamental group of the circle; intuitively this should be $\mathbb{Z}$, but proving this rigorously is surprisingly involved! To do so we will develop the theory of covering spaces.

Definition 2.10. A **covering space** of a topological space X is:

- a space $\widetilde{X}$ together with a map $p\colon \widetilde{X} \to X$ such that:
- each point $x \in X$ has a neighborhood $U \subset X$ so whose preimage $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by p.

- Such as set U is said to be **evenly covered**.
- Such a map p is called a **covering map**.

Terminology: A **lift** of a map $f\colon Y \to X$ to $p\colon \widehat{X} \to X$ is a map $\widehat{X}$ is a map $\widehat{f}\colon Y \to \widehat{X}$ so that $f = p\widehat{f}$.

$$\begin{tikzcd} & \widehat{X} \arrow[d, "\widehat{p}"] \\ Y \arrow[ur, dashed, "\widehat{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

A crucial feature of covering spaces is the following:

Definition 2.11. A map $p\colon \widehat{X} \to X$ has the **homotopy lifting property** if for any homotopy $f_t\colon Y \to X$ and any map $\widehat{f}_0\colon Y \to \widehat{X}$ lifting f_0, there exists a unique homotopy $\widehat{f}_t\colon Y \to \widehat{X}$ that lifts f_t.

> Equivalently: Given $F\colon Y \times I \to X$, any lift $\widehat{F}\colon Y \times \{0\} \to \widehat{X}$ of $F|_{Y\times\{0\}}$ can be extended to a unique lift $\widehat{F}\colon Y \times I \to \widehat{X}$ of F.

Lemma 2.12. *The homotopy lifting property implies **unique path lifting**: For each path $f\colon I \to X$ starting at $f(0) = x \in X$ and each $\widehat{x} \in p^{-1}(x)$, there is a unique lift $\widehat{f}\colon I \to \widehat{X}$ starting at $\widehat{x}$.*

Proof. 1) The path f is a homotopy $f\colon * \times I \to X$, hence given a lift $* \mapsto \widehat{x}$ of $* \mapsto x$, we can lift to a unique homotopy $\widehat{f}\colon \{*\} \times I \to \widehat{X}$. □

Proposition 2.13. *Every covering map $p\colon \widetilde{X} \to X$ satisfies the homotopy lifting property.*

Proof. Consider $F\colon Y \times I \to X$ and a lift $\widetilde{F}\colon Y \to \widetilde{X}$ of $F|_{Y\times\{0\}}$. Given a point $y_0 \in Y$, let us first build a lift $\widetilde{F}\colon N \times I \to \widetilde{X}$ for some neighborhood N of y_0. Each $F(y_0, t)$ has an evenly covered nbhd U_t, and by continuity we may find a product nbhd $N_t \times (a_t, b_t) \subset F^{-1}(U_t)$ of (y_0, t). Since $\{y_0\} \times I$ is compact, we can take a single nbhd N of y_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I so that each $F(N \times [t_i, t_{i+1}]) \subset U_i$ for some some evenly covered nbhd U_i.

Assume inductively $\widetilde{F}$ has been constructed on $N \times [0, t_i]$, which we are given for $i = 0$. Since U_i is evenly covered, there is a nbhd $\widetilde{U}_i \subset \widetilde{X}$ of $\widetilde{F}(y_0, t_i)$ that projects homeomorphically onto U_i. Shrinking N we may assume $\widetilde{F}(N \times \{t_i\}) \subset \widetilde{U}_i$. Then may define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ by composing F with $p^{-1}\colon U_i \to \widetilde{U_i}$. By the gluing lemma, this lets us extend $\widetilde{F}$ to $N \times [0, t_{i+1}]$. By induction, we obtain our lift $\widetilde{F}\colon N \to \widetilde{X}$.

Stop: 1/20/2026

To conclude, we argue any two such lifts $\widetilde{F}_j\colon N_j \times I \to \widetilde{X}$ agree on the common segment $\{y\} \times I$ for any $y \in N_1 \cap N_2$. This will prove our maps $N \times I \to \widetilde{X}$ glue together to give a full lift $Y \times I \to X$ and moreover that the lift is unique.

As before, may partition $0 = t_0 < t_1 < \cdots < t_m = 1$ so that $F(\{y\} \times [t_i, t_{i+1}])$ is contained in an evenly covered nbhd U_i of $F(y, t_i)$. Inductively assume $\widetilde{F}_1 = \widetilde{F}_2$ on $\{y\} \times [0, t_i]$. Now, the sets $F_j(\{y\} \times [t_i, t_{i+1}])$ are both connected and hence each must be contained in a single one of the disjoint sets in $p^{-1}(U_i)$ projecting homeomorphically to U_i; in fact these must in the same set $\widetilde{U_i}$, namely the unique one containing $F_1(y, t_i) = F_2(y, t_i)$. Since p is injective on $\widetilde{U_i}$ and $p\widetilde{F}_1 = p\widetilde{F}_2$, it follows that $\widetilde{F}_1 = \widetilde{F}_2$ on $[t_i, t_{i+1}]$, completing the induction. □

Note the following key consequence of unique path lifting:

Lemma 2.14. *Suppose $p\colon \hat{X} \to X$ has the homotopy lifting property. If $f_t\colon I \to X$ is a homotopy of paths (meaning the endpoints are fixed during the homotopy) then any lift $\hat{f}_t\colon I \to \hat{X}$ is also a homotopy of paths.*

Note that one can always find a lift of the homotopy: Choose any point $x \in p^{-1}(f_0(0))$. Then f_0 lifts to a unique path $\hat{f}_0$ at x. Then f_t lifts to a unique homotopy $\hat{f}_t$ of $\hat{f}_0$

Proof. The map $t \mapsto \hat{f}_t(0)$ defines a path $I \to \hat{X}$ that lifts the path $t \mapsto f_t(0)$, which is a constant path. We may always lift a constnat path to a constnat path, it follows from unique path lifting that $f \mapsto \hat{f}_t(0)$ in fact must be a constant path. Similarly $t \mapsto \hat{f}_1(t)$ must also be constant, because it is a lift of the constant path $t \mapsto f_t(1)$. Therefore $\hat{f}_t$ is indeed a homotopy of paths. □

2.2 *The fundamental group of the circle*

With the tool of covering spaces in hand, we can now calculate our first real example:

Theorem 2.15. *The fundamental group $\pi_1(S^1)$ of the circle is the infinite cyclic group $\mathbb{Z}$, generated by the homotopy class of the loop*

$$\omega(s) = e^{2\pi i s} = (\cos(2\pi s), \sin(2\pi s)) \quad \textit{based at } (1,0).$$

Proof. Note that $[\omega]^n = [\omega_n]$ where $\omega_n(s) = e^{n2\pi i s}$. Thus our task is to show that every loop in S^1 based at $(1,0)$ is homotopic to w_n for some unique $n \in \mathbb{Z}$.

Recall that map $p\colon \mathbb{R} \to S^1$ given by $p(s) = e^{2\pi i s}$. This is a covering map, since each $x = e^{2\pi i s} \in S^1$ has a nbhd $U_x = S^1 \setminus \{-x\}$ whose preimage is a disjoint union $\bigcup_{n \in \mathbb{Z}} (s + n - \pi, s + n - \pi)$ of open sets mapping homeomorphically to U_x.

Consider any loop $f\colon I \to S^1$ based at $x_0 = (1,0) = e^{i0}$. By the homotopy lifing property, there is a lift $\widetilde{f}\colon I \to \mathbb{R}$ starting at $\widetilde{f}(0) = 0$. This ends at some integer point $\widetilde{f}(1) \in p^{-1}(0) = \mathbb{Z}$. Say $\widetilde{f} = n$. Consider the path $\widetilde{\omega}_n\colon I \to \mathbb{R}$ given by $\widetilde{\omega}_n(s) = ns$. Observe that:

- The paths $\widetilde{f}, \widetilde{w}_n \colon I \to \mathbb{R}$ have the same endpoints and are therefore homotopic by a linear homotopy! ($(1-t)\widetilde{f} + t\widetilde{\omega}_n$).
- Composing this homotopy with p gives a homotopy from $f = p\widetilde{f}$ to $\omega_n = p\widetilde{\omega}_n$. Thus $[f] = [\omega_n] = [\omega]^n$.

It remains to show n is uniquely determined by f. So suppose $\omega_n \simeq f \simeq w_m$, meaning we have a homotopy h_t from $\omega_m = h_0$ to $\omega_n = h_1$. By the homotopy lifting property, this lifts to a homotopy $\widetilde{h}_t \colon I \to \mathbb{R}$ between $\widetilde{\omega}_n = \widetilde{h}_0$ and $\widetilde{\omega}_m = \widetilde{h}_0$. This must be a homotopy of paths, which means $\widetilde{\omega}_n$ and $\widetilde{\omega}_m$ have the same endpoints. Since $\widetilde{\omega}_n$ is a path from 0 to n, and $\widetilde{\omega}_m$ a path from 0 to m, it follows that $m = n$. □

Applications

There are several remarkable applications of the fact $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

Theorem 2.16 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. After rescaling, we ay write $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. If there are no loops, we may define

$$f_r(s) = \frac{p(re^{2\pi i s})}{\left|p(re^{2\pi i s})\right|}\frac{|p(r)|}{p(r)}$$

For each $r \geq 0$ this gives a loop in $\mathbb{S}^1 \subset \mathbb{C}$ based at 1. Varying r gives a homotopy of loops. Since f_0 is the trivial loop, we conclude $[f_r]$ is the trivial element of $\pi_1(S^1)$ for each r.

Now fix $r > 1 + |a_1| + \cdots + |a_n|$. Then for $|z| = r$ we have

$$\begin{aligned} |z|^n &> (|a_1| + \cdots + |a_n|)\left|z^{n-1}\right| \\ &\geq \left|a_1 z^{n-1}\right| + \cdots + |a_1| \geq \left|a_1 z^{n-1} + \cdots + a_n\right| \end{aligned}$$

Thus $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no roots on the circle $|z| = r$ when $0 \leq t \leq 1$. Replacing p with p_t in formula gives a homotopy from f_r (when $t = 1$) to the loop $\omega_n(s) = e^{2\pi i n s}$ (when $t = 0$). But we have seen $[\omega_n]$ is nontrivial in $\pi_1(\mathbb{S}^1)$! □

Theorem 2.17 (Brouwer fixed point theorem)**.** *Every continuous map h from the unit disk D^2 to itself has a fixed point.*

Proof. Suppose h has no fixed point, that is $h(x) \neq x$ for all $x \in D^2$. Define $r \colon D^2 \to \mathbb{S}^1$ by sending x to the point where the ray from $h(x)$ to x hits $\mathbb{S}^1$. Then r is a continuous. And clearly $r(x) = x$ for all $x \in \mathbb{S}^1$. That is, r is a retraction from D^2 onto S^1.

Now, for any loop f_0 in $\mathbb{S}^1$, there is a homotopy f_t in D^2 from f_0 to a constant loop. Then rf_t is a homotopy in $\mathbb{S}^1$ from f_0 to a constant loop. This contradicts that the group $\pi(\mathbb{S}^1)$ is nontrivial. □

We also mention the following interesting consequence, the proof of which can be found in the textbook:

Theorem 2.18 (Borsuk–Ulam Theorem)**.** *For every continuous map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ there are antipodal points with $f(x) = f(-x)$.*

Note that the $n = 1$ case is is an easy application of the intermediate value theorem, since $f(x) - f(-x)$ must change sign as x goes halfway around the circle. We'll prove the $n \geq 3$ cases later on using homology

For example, at any moment there must be a pair of antipodal points on the surface of the earth with the same temperature and pressure.

Stop: 1/22/2026

Start: 2/3/2026

Fundamental groups of products

Theorem 2.19. *If X and Y are path connected, then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

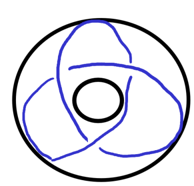
Proof. The universal property of products gives, for any space Z, a correspondence

$$\Big\{\text{maps } Z \to X \times Y\Big\} \quad \leftrightsquigarrow \quad \Big\{\text{pairs of maps } Z \to X \text{ and } Z \to Y\Big\}.$$

Thus, a loop f in $X \times Y$ based at (x_0, y_0) is equivalent the pair of a loop g in X based at x_0 and a loop h in Y based at y_0. Moreover, a homotopy f_t of loop in $X \times Y$ is equivalent to a homotopy g_t of loops in X and a homotopy h_t of loops in Y. This gives a bijection $[f] \mapsto ([g], [h])$ which is easily seen to be a group homomorphism. □

Example 2.20 (The torus)**.** We have $\pi_1(T^2) = \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \times \mathbb{Z}$. Under this isomorphism $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to a loop that winds p times around the first factor of $\mathbb{S}^1$ and q times around the second.

(2,3) curve on $T^2 = S^1 \times S^1$



2.3 Functoriality

So far we have associated a group to each pointed topological space; we next need to associate homomorphisms to maps between pointed spaces.

Proposition 2.21. *Any continuous map $\varphi\colon X \to Y$ induces a group homomorphism*

$$\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0)), \quad \textit{given by} \quad \varphi_*[f] = [\varphi f]$$

defined by post-composing loops based at x_0 with φ.

Proof.
- **well-defined** since if f_t is a homotopy of loops $I \to X$ at x_0, then φf_t is a homotopy of loops $I \to Y$ at $\varphi(x_0)$.
- **homomorphism:** since if f and g are loops at x_0, then $\varphi(f \cdot g) = (\varphi f) \cdot (\varphi g)$ as loops at y_0. □

Definition 2.22. A **pointed space** (X, x_0) is a topological space X with a designated basebpoint $x_0 \in X$. A morphism $(X, x_0) \to (Y, y_0)$ between pointed spaces is a continuous map taking x_0 to y_0. These form the **category of pointed spaces** $\mathrm{Top}_\bullet$.

Proposition 2.23. *π_1 is a functor from $\mathrm{Top}_\bullet$ to Grp.*

Proof. We've associated map $\varphi\colon (X, x_0) \to (Y, y_0)$ a homomorphism $\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ between their fundamental groups. It is immediate that $\mathbb{1}_* = \mathbb{1}$, and easy to see that $(\varphi\psi)_* = \varphi_*\psi_*$ due to associativity of composition $(\varphi\psi)f = \varphi(\psi f)$. □

For example, here is a simple application:

Proposition 2.24. *If $r\colon X \to A$ is a retract onto a subspace and $\iota\colon A \to X$ is the inclusion, then $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.*

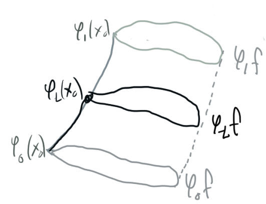
Proof. Since $r\iota = \mathbb{1}_A$, we have $\mathbb{1} = r_*\iota_*$ showing ι_* is injective. □

This gives another proof that there is no retraction $D^2 \to \mathbb{S}^1$ (the key ingredient in the Brouwer fixed point theorem), since the inclusion induced map $\mathbb{Z} = \pi_1(\mathbb{S}^1) \to \pi_1(D^2) = 0$ cannot be injective.

The next thing to show is that homotopic maps induces the same group homomorphism. Since homotopies can move the basepoint, this requires a small bit of care.

Lemma 2.25. *If $\varphi_t\colon X \to Y$ is a homotopy and h is the path $t \mapsto \varphi_t(x_0)$ in Y, then $\varphi_{0*} = \beta_h \varphi_{1*}$. In particular, if φ_t is basepoint-preserving, then h is the constant path and $\varphi_{0*} = \varphi_{1*}$.*

$$\begin{tikzcd}
 & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \\
 & \pi_1(Y, \varphi_0(x_0))
\end{tikzcd}$$



Proof. Let $h_t(s) = h(ts)$, so that h_t is a path from $h(0) = \varphi_0(x_0)$ to $h(t)$. Then if f is a loop in X based at x_0, then $h_t \cdot (\varphi_t \cdot f)\overline{h}_t$ is a homotopy of loops in Y based at $\varphi_0(x_0)$, interpolating between $\varphi_0 f$ and $h \cdot \varphi_1 f \cdot \overline{h} = \beta_h(\varphi_1 f)$. □

Theorem 2.26. *If $\varphi\colon X \to Y$ is a homotopy equivalence, then $\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism.*

in particular, if X deformation retracts onto a subspace A, then $\pi_1(X) \cong \pi_1(A)$.

Proof. Let $\psi\colon Y \to X$ be a homotopy inverse. Then $\psi\varphi \simeq \mathbb{1}_X$ which implies $\psi_*\varphi_* = \beta_h$ for some path in X. In particular, since β_h is an isomorphism, it follows that ψ_* is surjective and φ_* injective. Similarly $\varphi\psi \simeq \mathbb{1}_Y$ implies φ_* is surjective and ψ_* is injective. □

3 Van Kampen's Theorem

This is a powerful tool that gives a method of computing the fundamental groups by decomposing spaces into simpler pieces whose fundamental groups are already known.

3.1 Free products of groups

Given a collection of groups G_α this is a way to produce a new group that contains these as subgroups. One option would be to take the direct product, but that is unnatural in our context of nonabelian groups.

Definition 3.1. The **free product** $*_\alpha G_\alpha$ consists of:

- reduced words $g_1 g_2 \cdots g_m$ of finite length, where each g_i belongs to some group G_{α_i} and where reduced means:
  - each g_i not the trivial element
  - adjacent letters g_i and g_{i+1} belong to distinct groups
- any unreduced word can be simplified to a reduced one by combining adjacent letters into one and canceling identity elements.
- the group operation is: concatenate and simplify to a reduced word. Inverses are given by reversing the order and inverting elements:

$$(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \ldots h_n \rightsquigarrow \text{ simplify and reduce}$$
$$(g_1 \cdots g_m)^{-1} = g_m^{-1} \cdots g_1^{-1}$$

One needs to show the resulting reduced word is independent of the choices made in reducing!

Exercise 3.2. Verify that this indeed defines a group. (Nontrivial to show its associative!)

Fact 3.3. *The free product $G = *_\alpha G_\alpha$ is characterized by the following universal property in the category* Grp*: There exists morphisms $\iota_\alpha \colon G_\alpha \to G$ such that for any group H and morphisms $f_\alpha \colon G_\alpha \to H$, there exists a unique morphism $f \colon G \to H$ so that the maps f_α factor as $f_\alpha = f\iota_\alpha$.*

This the categorical **coproduct**

$$\begin{tikzcd}
G_\alpha \arrow[dr, "\iota_\alpha"'] \arrow[drr, "f_\alpha"] & & \\
\vdots & G \arrow[r, dashed, "f"] & H \\
G_\beta \arrow[ur, "\iota_\beta"] \arrow[urr, "f_\beta"'] & &
\end{tikzcd}$$

Exercise 3.4. Verify the following:

- the construction of $*_\alpha G_\alpha$ satisfies the universal property
- the universal property determines $*_\alpha G_\alpha$ up to isomorphism
- each morphism $\iota_\alpha \colon G_\alpha \to G$ is in fact injective; which means we may view G_α as a subgroup of G.

3.2 *Variation: Amalgamated free products*

Definition 3.5. Any pair of group morphism $\varphi_j \colon K \to G_j$ determine a **push out** or **amalgamated free product** $G = G_1 *_K G_2$ defined by the following universal property: There are maps $\iota_j \colon G_j \to G$ satisfying $\iota_1\varphi_1 = \iota_2\varphi_2$ such that for any group H and morphism $f_j \colon G_j \to H$ satisfying $f_1\varphi_1 = f_2\varphi_2$, there is a unique morphism $f \colon G \to H$ so that $f_j = f\iota_j$.

$$\begin{tikzcd}
& G_1 \arrow[dr, "\iota_1"'] \arrow[drr, "f_2"] & & \\
K \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & G \arrow[r, dashed] & H \\
& G_2 \arrow[ur, "\iota_2"] \arrow[urr, "f_1"'] & &
\end{tikzcd}$$

Exercise 3.6. Show the free product and amalgamated free product are related as follows:

$$G_1 *_K *G_2 = (G_1 * G_2)/N,$$

where $N \lhd G_1 * G_2$ is the normal subgroup generated by all elements of the form $(\iota_1\varphi_1(k))(\iota_2\varphi_2(k))^{-1}$.

3.3 The van Kampen Theorem

Theorem 3.7 (Van Kampen Theorem). *Suppose X is the union of path-connected open subset A_α, each containing the basepoint $x_0 \in X$. Then the induced maps $j_\alpha \colon \pi_1(A_\alpha) \to \pi_1(X)$ from the inclusions $A_\alpha \hookrightarrow X$ determine a homomorphism $\Phi \colon *_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$.*

- *If each $A_\alpha \cap A_\beta$ is path connected, then Φ is surjective.*
- *If each $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\ker(\Phi)$ is the normal subgroup N generated by all elements of the form $\iota_{\alpha\beta}(\omega)\iota_{\beta\alpha}(\omega)^{-1}$. (Here $\iota_{\alpha\beta}$ is the map $\pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ induced by inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.)*

Special case: Suppose $X = A \cup B$ where $A, B \subset X$ are open, path-connected and $C = A \cap B$ is path-connected. Then the condition on triple intersections is vacuous! and the fundamental group is the amalgamated free product:

$$\pi_1(X) = \pi_1(A) *_{\pi_1(C)} \pi_1(B).$$

Before doing the proof, lets consider some applications:

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Theorem 3.8. *$\pi_1(\mathbb{S}^n) = 0$ for $n \geq 2$.*

Proof. Let $A, B \cong \mathbb{R}^n$ open neighborhoods of the northern and southern hemispheres, so $\mathbb{S}^n = A \cup B$ with $A \cap B \cong \mathbb{S}^{n-1} \times \mathbb{R}$. Taking the basepoint $x_0 \in A \cap B$, we get $\pi_1(A) = \pi_1(B) = 0$, so $\pi_1(\mathbb{S}^n) = \pi_1(A) *_{\pi_1(A\cap B)} \pi_1(B) = 0$. □

❓ What's wrong with this proof in the case $n = 1$?

🔓 Then $A \cap B \cong S^0 \times \mathbb{R}$ is not path-connected! This is why we can't use van Kampen to calculate $\pi_1(\mathbb{S}^1)$ and must instead use covering spaces

Corollary 3.9. *$\mathbb{R}^2$ is not homeoomrphic to $\mathbb{R}^n$ for $n \neq 2$.*

Proof. Any homeomorphism $\mathbb{R}^n \to \mathbb{R}^2$ would gives a homeomorphism $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^2 \setminus \{f(0)\}$. But these spaces have different fundamental groups since they are respectively homotopy equivalent to $\mathbb{S}^{n-1}$ and $\mathbb{S}^1$. □

Example 3.10. Consider a wedge sum $X = \bigvee_\alpha X_\alpha$ of path-connected spaces. Assume each basepoint $x_\alpha \in X_\alpha$ is a deformation retraction of an open nbhd U_α in X_α. Then X_α is a deformation retraction of its nbhd $A_\alpha = X_\alpha \bigvee_{\beta\neq\alpha} U_\beta$ in X. The intersection of two or more A_α's is $\bigvee_\alpha U_\alpha$, which is path connected and contractible. Thus van Kampen's theorem says the map $*_\alpha \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$ is an isomorphism.

- Eg. $\pi_1\left(\bigvee_\alpha \mathbb{S}^1\right) = *_\alpha\mathbb{Z}$ is a free group.

Definition 3.11. The **free group** on a set A is the free product

$$\mathbb{F}(A) = *_A\mathbb{Z}$$

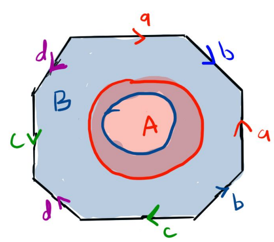
of cyclic groups $\mathbb{Z} = \langle a\rangle$, one for each letter $a \in A$. If $\mathcal{R}$ is a set of words in the letters of A and their inverses, we may form the **group presentation** as the quotient

$$\langle A \mid \mathcal{R}\rangle := *_A\mathbb{Z}/\langle\langle\mathcal{R}\rangle\rangle$$

of the free group by the normal closure of $\mathcal{R}$ (the smallest normal subgroup containing $\mathcal{R}$).

Example 3.12. Recall that the closed surface M_g of genus g is a quotient of a $4g$–gon by pairwise identifying sides. Let A be the (image of) interior of the polygon, and B (the image of) a nbhd of the Boundary. Then $A, B, A \cap B \subset M_g$ are open path-connected subsets with:

$$A \text{ contractible } \rightsquigarrow \pi_1(A) = 0$$
$$B \simeq \bigvee_{i=1}^{2g} \mathbb{S}^1 \rightsquigarrow \pi_1(B) = \mathbb{F}(a_1, b_1, \ldots, a_g, b_g)$$
$$A \cap B \simeq \mathbb{S}^1 \rightsquigarrow \pi_1(A \cap B) = \mathbb{Z}$$



$\pi_1(A \cap B) = a b \bar{a} \bar{b} c d \bar{c} \bar{d}$

The generator of $\pi_1(A \cap B)$ is trivial in A and maps to the product of commutators $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ in $\pi_1(B)$, so

$$\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g]\rangle$$

Corollary 3.13. *Surfaces M_g and M_h are not homotopy equivalent if $g \neq h$.*

Proof. $\pi_1(M_g) \neq \pi_1(M_h)$ since they have different abelianizations $\mathbb{Z}^{2g}$ and $\mathbb{Z}^{2h}$. ☐

Proof of van Kampen Thorem.

By a factorization of an element $[f] \in \pi_1(X)$, mean a decomposition $[f] = [f_1] \cdots [f_m]$ as a formal product where each f_i is a loop at x_0 in some A_α. We say two factorizations are equivalent if they differ by a sequence of moves or their inverses:

We skip the proof in lecture in the interest of time (we lost a week due to ice storm)

that is, a factorization as a word in $*_\alpha\pi_1(A_\alpha)$.

- Replace $[f_i][f_{i+1}]$ by $[f_i \cdot f_{i+1}]$ if both loops in same A_α.

- If f_i a loop in $A_\alpha \cap A_\beta$, then swap between regarding $[f_i]$ as an elt of $\pi_1(A_\alpha)$ vs an elt of $\pi_1(A_\beta)$.

The first move does not change the element of $*_\alpha \pi_1(A_\alpha)$, and the second does not change its image in the quotient $Q = *_\alpha \pi_1(A_\alpha)/N$.

Proving surjectivity amounts to showing each $[f] \in \pi_1(X)$ has a factorization, and injectivity amounts two showing any two factorizations of $[f]$ are equivalent, as this will prove $Q \to \pi_1(X)$ is injective.

and thus that the kernel of $\Phi\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is exactly N

Surjectivity: Let $f\colon I \to X$ be any loop based at x_0. Since each $f^{-1}(A_\alpha)$ is open and I is compact, we may cover I by finitely many intervals $[a_i, b_i]$ each contained in some $f^{-1}(A_\alpha)$. Taking all endpoints of these intervals gives a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ with each $f([s_i, s_{i+1}])$ is contained in some A_α, which we denote A_i.

View f as a concatenation $f = f_1 \cdots f_m$, where $f_i = f|_{[s_i, s_{i+1}]}$. For $0 < s < m$, since $f(s_i) \in A_i \cap A_{i+1}$, which is path connected, we may fix a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i)$. Then

$$f = f_1 \cdots f_m \simeq (f_1 \cdot \overline{g_1}) \cdot (g_1 \cdot f_2 \cdot \overline{g_2}) \cdot \cdots \cdot (g_{m-1} \cdot f_m)$$

gives a factorization of $[f]$ into loops that are each in a single A_i.

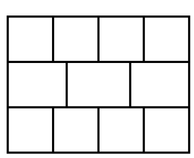
Injectivity: Suppose $[f_1] \dots [f_m]$ and $[f_1'] \dots [f_n']$ are two factorizations of $[f]$. Then there is a homotopy $F\colon I \times I \to X$ between the concatenations $f_1 \cdots f_m$ and $f_1' \cdots f_n'$. Again (since $I \times I$ compact and each $F^{-1}(A_\alpha)$ open), may partition $I \times I$ into a "brick pattern" of rectangles R_j so that:

- Each $F(R_j)$ contained in a single $A_\alpha = A_j$

- Each point of $I \times I$ contained in at most 3 rectangles

- concatenation points of $f_1 \cdots f_k$ on bottom and $f_1' \cdots f_\ell'$ on top are vertices of rectangles

the top and bottom may have more vertices than these!



Since any vertex v lies in at most 3 rectangles R_i, R_j, R_k, we may fix a path g_v from x_0 to $F(v)$ in the path-connected set $A_i \cap A_j \cap A_k$. If already $F(v) = x_0$, take g_v to be constant path. Now:

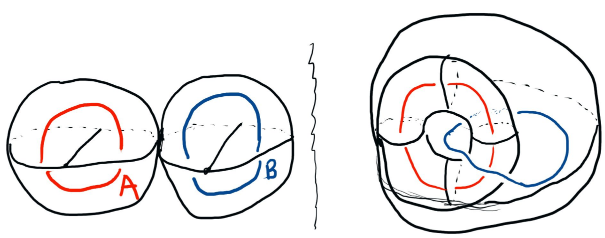
- Every edge e from v to w our partition now gives a loop $\gamma_e = g_v \cdot F|_e \cdot \overline{g_w}$ in some A_i based at x_0.

- Any edge path $e_1 e_2 \cdots e_k$ thus determines a concatenation of loops, factorized as a word $\gamma_{e_1} \cdot \gamma_{e_2} \cdots \gamma_{e_k}$ in $*_\alpha \pi_1(A_\alpha)$!
- If $e \in R_i \cap R_j$, then γ_e is a loop in $A_i \cap A_j$!
  - swapping from viewing $e \in R_i$ vs $e \in R_j$ applies a type–2 move to our factorization!
- Perimeter of rectangle R_i gives nullhomotopic loop in A_i!
  - Thus going one way vs the other around R_i changes factorization by a type–1 move

This gives a sequence of moves between our two factorizations $f_1 \cdots f_m$ and $f'_1 \cdots f'_n$: Indeed these differ from the edge paths on the bottom and top of $I \times I$ type 1 moves. And we can go from bottom edge path to top, by sequentially "changing edge allegiances" and pushing across the rectangles R_i. □

Example 3.14 (Linking circles)**.** Let A be an embedded circle in $\mathbb{R}^3$. It's not hard to see that $\mathbb{R}^3 \setminus A$ deformation retracts onto the union of a sphere $\mathbb{S}^2$ with a diameter, which is homotopy equivalent to $\mathbb{S}^2 \vee \mathbb{S}^1$. Thus $\pi_1(\mathbb{R}^3 \setminus A) = \mathbb{Z}$. Now if B is another circle that is unlinked from A, then similarly $\mathbb{R}^3 \setminus (A \cup B)$ deformation retracts on to $(\mathbb{S}^2 \vee \mathbb{S}^1) \vee (\mathbb{S}^2 \vee \mathbb{S}^1)$. Thus $\pi_1(\mathbb{R}^3 \setminus (A \cup B)) = \mathbb{Z} * \mathbb{Z}$. However, if A and B are linked, then $R^3 \setminus (A \cup B)$ deformation retracts onto the wedge of $\mathbb{S}^2 \vee (\mathbb{S}^1 \times \mathbb{S}^1)$ and has fundamental group $\mathbb{Z} \times \mathbb{Z}$.

Note that we skipped this example in lecture



One important consequence of van Kampen's theorem is that is lets you understand the effect of attaching cells to CW complexes:

Proposition 3.15. *If Y is obtained from X by attaching cells of dimension at least 2, then $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is the normal subgroup N generated by the conjugacy classes of the loops given by the gluing maps $\varphi_\alpha \colon S^1 \to X$ **of the 2–cells**! In particular:*

- *Attaching cells of dimension $n > 2$ does not affect π_1!*
- *The inclusion $X^2 \hookrightarrow X$ of the 2–skeleton induces an isomorphism on π_1.*

Recall from the HW that every unbased loop in X determines a conjugacy class in $\pi_1(X, x_0)$. And since we're talking about a normal subgroup here, it contains the whole conjugacy class

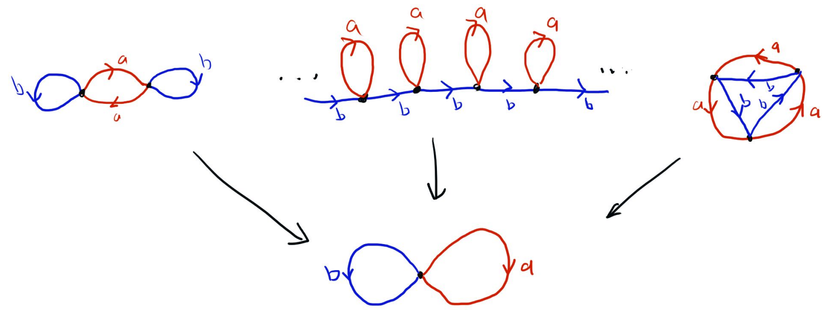
Corollary 3.16. *For every group G, there is a 2–dimensional complex X_G with $\pi_1(X_G) \cong G$.*

Proof. Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$ (always exists since every group is a quotient of a free group). Then build X_G from $\bigvee_\alpha \mathbb{S}^1_\alpha$ by attaching 2–cells e^2_β by the loops specified by the words r_β. $\square$

4 *Covering Spaces*

We now want to try and understand the different possible covering spaces $\widetilde{X}$ of a give space X.

Example 4.1. Here are some of the covering spaces of the wedge $X = \mathbb{S}^1 \vee \mathbb{S}^1$ of two circles:



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Recall that covering spaces have the homotopy lifting property. In particular, paths have unique lifts, and thus constant paths lift to constant paths! This has many important consequences:

Proposition 4.2. *Any covering map* $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ *induces an injective map* $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$*. The image subgroup* $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ *consists of the homotopy classes of loops in* X *at* x_0 *whose lifts to* $\widetilde{X}$ *at* $\widetilde{x}_0$ *are loops.*

Proof. Suppose $\widetilde{f}_0\colon I \to \widetilde{X}$ is a loop at $\widetilde{x}_0$ in the kernel. Then there's a homotopy f_t from $f_0 = p\widetilde{f}_0$ to the constant loop f_1. By the homotopy lifting property, this lifts to a homotopy $\widetilde{f}_t$ from $\widetilde{f}_0$ to the constant path $\widetilde{f}_1$, showing $[\widetilde{f}_0] = 0$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$.

The second statement characterizing $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is clear (since a loop upstairs maps to a loop that lifts, and a loop that lifts is clearly the image of that lift!) □

Recall that $U \subset X$ is **evenly covered** if $p^{-1}(U)$ is a disjoint union $\sqcup U_\alpha$ of open sets each mapping homeomorphically onto U. These sets U_α are called **sheets** of $\widetilde{X}$ over U. The **degree** or **number of sheets** of U is the cardinality of $p^{-1}(x)$ for any $x \in U$; this is locally constant, and hence globally constant if X is connected.

Proposition 4.3. *The degree of a cover* $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ *with* $\widetilde{X}, X$ *path-connected, equals the index of* $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ *in* $\pi_1(X, x_0)$.

Proof. Set $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Define a map $\Phi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ by sending a loop g in X at x_0 to the endpoint $\widetilde{g}(1)$ of a lift $\widetilde{g}$ of g starting at x_0. This is well-defined, since homotopic loops lift to paths with the same endpoints (a htpy between them lifts to a htpy of paths). This map is surjective, since any $y \in p^{-1}(x_0)$ can be joined to x_0 by a path $\widetilde{g}$ projecting to a loop at x_0 lifting to $\widetilde{g}$. And

$$\begin{aligned}\Phi([g_1]) = \Phi([g_2]) &\iff \widetilde{g}_1(1) = \widetilde{g}_2(1) \\ &\iff \widetilde{g}_1 \cdot \overline{\widetilde{g}_2} \text{ is a loop in } \widetilde{X} \text{ at } \widetilde{x}_0 \\ &\iff [g_1][g_2]^{-1} \in H\end{aligned}$$

So Φ gives a bijection between $p^{-1}(x_0)$ and the left cosets $H[g]$ of H. $\square$

In fact, one can lift general maps, not just homotopies. When this is possible is dictated the following **lifting criterion** in terms of the image subgroup of the covering:

Proposition 4.4. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering and let Y be path-connected and locally path-connected space.*
Then a map $f\colon (Y, y_0) \to (X, x_0)$ admits a lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ if and only if $f_*(\pi_1(Y, y_0)) \le p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Proof. The "only if" is immediate from functoriality since $f_* = p_*\widetilde{f}_*$. For the converse, we construct the lift $\widetilde{f}$: Given $y \in Y$, let γ be a path from y_0 to y. Then $g = f\gamma$ is a path in X starting at x_0 which we may lift to a path $\widetilde{g}$ in $\widetilde{X}$ starting at $\widetilde{x}_0$. Define $\widetilde{f}(y) = \widetilde{g}(1)$.

This is well-defined, since if γ' is another path from y_0 to y, then $h = (f\gamma') \cdot \overline{(f\gamma)}$ is a loop at x_0 with $[h] \in f_*(\pi_1(Y, y_0)) \le p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$; hence it lifts to a loop $\widetilde{h}$ in at $\widetilde{x}_0$. By uniqueness of path lifting, the first half of $\widetilde{h}$ must be $\widetilde{g}'$ and the second half must be $\widetilde{g}$ backwards. Thus $\widetilde{g}'(1) = \widetilde{g}(1)$.

Now use local path-connectedness to prove continuity: Take an evenly covered nbhd U of $f(y)$ and a path-connected nbhd V of y with $f(V) \subset U$. Then paths to points $y' \in V$ can be built by concatenating a fixed path γ to y with a path η from y to y' in V. If $\widetilde{U}$ is the sheet over U containing $\widetilde{f}(Y)$, it follows that $\widetilde{f}|_V$ is given by composing f with $p^{-1}\colon U \to \widetilde{U}$, which is continuous. $\square$

We also have the **unique lifting property**:

Proposition 4.5. *If $p\colon \widetilde{X} \to X$ is a covering map and $f\colon Y \to X$ a map with Y connected. Then any two lifts $\widetilde{f}_1, \widetilde{f}_2$ that agree at one point agree on all of Y.*

Proof. If $U \subset X$ is evenly covered and $\widetilde{U}$ is the sheet containing $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, then by continuity we may find a nbhd V of y mapping into $\widetilde{U}$ under both $\widetilde{f}_1$ and $\widetilde{f}_2$. In particular $f(V) \subset U$ and on V both maps $\widetilde{f}_i$ look like $V \xrightarrow{f} U \xrightarrow{p^{-1}} \widetilde{U}$ and hence agree. Thus the set where the maps agree is open and also closed (as the preimage of the diagonal under $\widetilde{f}_1 \times \widetilde{f}_2$). $\square$

Note that this basically follows from our proof above, since we didn't make any choices!

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In order to classify covering spaces of a given space, we need to decide when two covering spaces are "the same":

Definition 4.6. An **isomorphism** b/w covering spaces $p_1 \colon \widetilde{X}_1 \to X$ and $p_2 \colon \widetilde{X}_2 \to X$ of X is a homeomorphism $f \colon \widetilde{X}_1 \to \widetilde{X}_2$ so that $p_1 = p_2 f$:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Proposition 4.7. *Two covering spaces $p_i \colon (\widetilde{X}_i, \widetilde{x}_i) \to (X, x_0)$ of X admit a basepoint preserving isomorphism $f \colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ (assuming path-connected) iff the subgroups $(p_i)_*(\pi_1(\widetilde{X}_i, x_i))$ of $\pi_1(X, x_0)$ are equal.*

Proof. Clearly, the existence of a homeomorphism f with $p_1 = p_2 f$ implies the subgroups are equal. Conversely, if the subgroups are equal, the lifting criterion implies we may lift p_1 to $\widetilde{p}_1$ and lift p_2 to $\widetilde{p}_2$ so that $p_1 = p_2\widetilde{p}_1$ and $p_2 = p_1\widetilde{p}_2$. Further, uniqueness implies $\widetilde{p}_2\widetilde{p}_1 = \mathbb{1}_{\widetilde{X}_1}$ (since it lifts $\mathbb{1}_X$) and symmetrically $\widetilde{p}_2\widetilde{p}_1 = \mathbb{1}_{\widetilde{X}_2}$, so they are covering space isomorphisms. □

4.1 *The universal cover*

We've seen some features that all covering spaces have. Now we want to find all the covering spaces!

★ Main goal is to establish a correspondence between connected covering spaces of X and subgroups of $\pi_1(X)$!

Thus for every subgroup we need to build a covering space. In particular, for the trivial subgroup we need to build a cover with trivial π_1!

? What is the covering space associated the full subgroup $\pi_1(X)$?

The space X itself with the identity covering $X \to X$!

Definition 4.8. A simply-connected covering space of X is called a **universal cover**.

the relevance of the terminology "universal" will become apparent soon

To ensure the existence of such, we need a technical condition:

Definition 4.9. A space X is **semilocally simply-connected** if each point x has a nbhd U such that the inclusion induces a trivial map $\pi_1(U, x) \to \pi_1(X, x)$.

- We'll not worry about this; just note that CW complexes satisfy it

This is necessary, since if $p \colon \widetilde{X} \to X$ is a covering with $\widetilde{X}$ simply connected, then every evenly covered open set $U \subset X$ has this property, since loops in U lift to loops in $\widetilde{U}$ that are null-homotopic in $\widetilde{X}$

The idea of the construction is motivated by the observation that <u>if</u> we had a simply connected cover, then:

- each point of $\widetilde{X}$ is joined to $\widetilde{x}_0$ by a unique homotopy class of paths

because simply connected!

- homotopy lifting property gives correspondence between homotopy classes of paths in $\widetilde{X}$ at $\widetilde{x}_0$ and in X at x.

Definition 4.10. Let X be path-connected, locally path-connected, and semilocally simply-connected space with basepoint x_0. Its associated **universal cover** is the space of homotopy classes of paths:

$$\widetilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

along with the map $p\colon \widetilde{X} \to X$ sending $[\gamma]$ to $\gamma(1)$. Map is well-defined, since homotopies of γ don't change the endpoints, and surjective, since X is path-connected.

Topology on $\widetilde{X}$:

- Let $\mathcal{U}$ be the collection of path-connected open subsets $U \subset X$ so that $\pi_1(U) \to \pi_1(X)$ is trivial.
  - semilocally simply-connected $\implies$ a basis for topology on X!

key is that if $U \in \mathcal{U}$ and $V \subset U$ is open and path-connected, then $V \in \mathcal{U}$ also since $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is evidently trivial

- Given $U \in \mathcal{U}$ and $[\gamma] \in \widetilde{X}$, define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}$$

  - $p\colon U_{[\gamma]} \to U$ is bijective because $\pi_1(U) \to \pi_1(X)$ is trivial
  - ($\star$) If $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$.
  - This implies the sets $U_{[\gamma]}$ form a basis for a topology on $\widetilde{X}$.
  - Further, in this topology $p\colon U_{[\gamma]} \to U$ is a homeomorphism.

different choices of η in U joining $\gamma(1)$ to $x \in U$ are homotopic in X

Just need to check for subsets $V \subset U$ with $V \in \mathcal{U}$, for which we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma]}$.

- $p\colon \widetilde{X} \to X$ is a covering space, since for $U \in \mathcal{U}$ the sets $U_{[\gamma]}$ give a partition of $p^{-1}(U)$. ($U_{[\gamma]}$ and $U_{[\gamma']}$ are disjoint or equal).
- Finally, $\widetilde{X}$ is simply connected: Easy to see its path-connected, and $\pi_1(\widetilde{X}) = 0$ since its image under p_* (which is injective) is trivial.

By construction, a loop in X at x_0 lifts to a loop in $\widetilde{X}$ iff the loop is nullhomotopic in X

4.2 *Classification of covering spaces*

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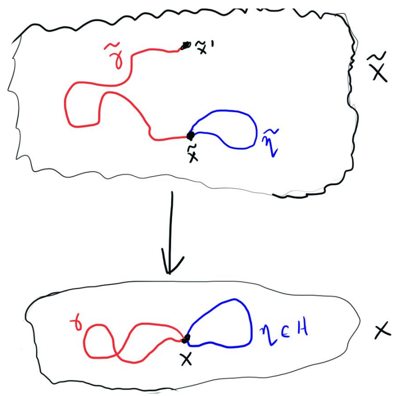
Theorem 4.11 (Fundamental Theorem of Covering spaces). *Let X be path-connected, locally path-connected, and semilocally simply connected. Then we have bijections:*

1. *basepoint-preserving isomorphism classes of path-connected covering spaces $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$.*

$$p \longleftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$$

2. *isomorphism classes of path-connected covering space* $\widetilde{X} \to X$ *and conjugacy classes of subgroups of* $\pi_1(X)$.

Proof. (2) follows from (1): Given a covering space $p\colon \widetilde{X} \to X$ and point $x \in X$, we may choose any $\tilde{x} \in p^{-1}(x)$ get a based covering $(\widetilde{X}, \tilde{x}) \to (X, x)$ and subgroup $H = p_*(\pi_1(\widetilde{X}, \tilde{x}))$ of $\pi_1(X, x)$. Changing to a new basepoint $\tilde{x}' \in p^{-1}(x)$ corresponds to conjugating H by an element $[\gamma]$, where $\tilde{\gamma}$ is a path from $\tilde{x}$ to $\tilde{x}'$ and $\gamma = p\tilde{\gamma}$. This is because a loop η at x lifts to a loop at $\tilde{x}$ iff $\overline{\gamma} \cdot \eta \cdot \gamma$ lifts to a loop at $\tilde{x}'$.



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Proof of (1): We already know each based covering gives a subgroup of $\pi_1(X, x)$ and that this assignment is injective. It remains to show that for each subgroup $H \leq \pi_1(X, x)$ there exists a based covering $(X_H, \tilde{x}) \to (X, x)$ with $p_*(\pi_1(X_H, \tilde{x})) = H$. We construct this from the universal covering $\widetilde{X}$! Define an equivalence relation on $\widetilde{X}$ by declaring $[\gamma'] \sim [\gamma]$ if $\gamma'(1) = \gamma(1)$ and $[\gamma' \cdot \overline{\gamma}] \in H$.

That this is an equivalence relation (reflective/symmetric/transtiive) follows from fact that H is a subgroup (has identity and closed under inverses/multiplication)!

- Requirement $\gamma'(1) = \gamma(1)$ implies $p\colon \widetilde{X} \to X$ descends to the quotient $X_H \to X$.
- This is in fact a covering, since $[\gamma'] \sim [\gamma]$ iff $[\gamma' \cdot \eta] \sim [\gamma \cdot \eta]$ implies that if any two points in basic nbhds $U_{[\gamma']}$ and $U_{[\gamma]}$ are identified, then the whole nbhds are identified.
- Finally, taking $\tilde{x}_0 \in X_H$ to be equivalence class of the constant path $[c]$ yields $p_*(\pi_1(X_H, \tilde{x}_0)) = H$, since a loop γ in X at x lifts to path in $\widetilde{X}$ from $[c]$ to $[\gamma]$ which descends to loop in X_H iff $[\gamma] \sim [c]$ iff $[\gamma] = [\gamma \cdot \overline{c}] \in H$. $\square$

4.3 *Deck Transformations*

Definition 4.12. A **deck transformation** is a self-isomorphism of a covering space $p\colon \widetilde{X} \to X$, that is, a homomorphism $f\colon \widetilde{X} \to \widetilde{X}$ so that $pf = p = p\mathbb{1}_X$.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & \widetilde{X} \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{\mathbb{1}_X} & X \end{array}$$

These form a group under composition, called the **deck group** $G(\widetilde{X})$.

If $\widetilde{X}$ is path connect then a deck transformation is determined by where it sends a single point (by unique lifting). Thus only the identity deck transformation fixes a point.

Definition 4.13. A covering space $p\colon \widetilde{X} \to X$ is **normal/regular** if for all $x \in X$ and all $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there exists $f \in G(\widetilde{X})$ with $f(\tilde{x}) = \tilde{x}'$.

These are the covering spaces with **maximal symmetry**

Theorem 4.14. *Let $p\colon \widetilde{X} \to X$ be a path-connected covering of a path-connected and locally path-connected space X and let $H = p_*(\pi_1(\widetilde{X}))$ be the associated subgroup (conjugacy class). Then:*

1. *The deck group $G(\widetilde{X})$ is isomorphic $N(H)/H$, where $N(H)$ is the normalizer of H.*

2. *The covering space is normal iff H is normal in $\pi_1(X, x)$.*

In particular, for the universal cover we have $G(\widetilde{X}) \cong \pi_1(X)$.

★ The identification $G(\widetilde{X}) \cong N(H)/H$ **depends on choosing basepoints** $\tilde{x} \in \widetilde{X}$ and $x = p(\tilde{x}) \in X$!!

Proof. Observe that:

$$\begin{aligned} &\text{there exists } \tau \in G(\widetilde{X}) \text{ sending } \tilde{x} \in p^{-1}(x) \text{ to } \tilde{x}' \in p^{-1}(x) \\ &\iff H = p_*(\pi_1(\widetilde{X}, \tilde{x})) \text{ equals } H' = p_*(\pi_1(\widetilde{X}, \tilde{x}')) \end{aligned}$$

This is because $p\tau = p = p\tau^{-1}$ implies $H = p_*(\tau_*(\pi_1(\widetilde{X}, \tilde{x})) \le H'$ and visaversa. Conversely, if $H = H'$ then the lifting criterion gives lifts τ, τ^{-1} of $\mathbb{1}_X$ sending $\tilde{x}$ to $\tilde{x}'$ and $\tilde{x}'$ to $\tilde{x}$.

We also have a correspondence between paths $\tilde{\gamma}$ from $\tilde{x}$ to $\tilde{x}'$ and projected loops $\gamma = p\tilde{\gamma}$ in $\pi_1(X, x)$ with $[\gamma]^{-1}H[\gamma] = H'$. Thus if $[\gamma] \in N(H)$ and we lift to the path $\tilde{\gamma}$ starting at x, then there is a unique deck transformation τ sending $\tilde{x}$ to $\tilde{\gamma}(1) \in p^{-1}(x)$. Thus we may define $\Phi\colon N(H) \to G(\widetilde{X})$ by $[\gamma] \mapsto \tau$.

- Φ is a homomorphism since if $\Phi([\gamma']) = \tau'$, then $\gamma \cdot \gamma'$ lifts to the path $\tilde{\gamma} \cdot \tau(\tilde{\gamma}')$ from $\tilde{x}$ to $\tau\tau'(\tilde{x})$.

- The above shows Φ is surjective. And $\ker(\Phi) = H$, since it consists precisely of classing $[\gamma]$ lifting to loops at $\tilde{x}$. Hence Φ yields an isomorphism $N(H)/H \xrightarrow{\cong} G(\widetilde{X})$.
- Now clear that H normal iff $G(\widetilde{X})$ acts transitively on $p^{-1}(x)$. □

The action of the deck group $G(\widetilde{X})$ on $\widetilde{X}$ satisfies the following:

Definition 4.15. A group action $G \curvearrowright Y$ on a topological space is a **covering space action** if every point $y \in Y$ has a nbhd V such that $g(V) \cap g'v(V) = \emptyset$ for all $g \neq g'$.

❓ Why does $G(\widetilde{X}) \curvearrowright \widetilde{X}$ satisfies this?

🔓 take the nbhd to be a sheet $\widetilde{U}$ over an evenly covered set $U \subset X$

Exercise 4.16. If Y is Hausdorff, this is equivalent to saying the action is free (only the identity fixes any point) and **properly discontinuous** (every point y has a nbhd so that $\{g \in G \mid U \cap g(U) \neq \emptyset\}$ is finite)

Proposition 4.17. *If $G \curvearrowright Y$ is a covering space action, then:*

- *The quotient $p\colon Y \to Y/G$ is a normal covering space and,*
- *G is the deck group, provided Y is path-connected.*
- *G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$ if Y path-connected and locally path-connected.*

Proof. Exercise □

💬 In particular, if Y is simply connected and locally path-connected, then $G \cong \pi_1(Y/G)$. This gives a useful technique to calculate the fundamental group of a space: Build a simply-connected cover and find its deck group!

Remark 4.18. If $H \leq G$ we have an intermediate quotient

$$Y \to Y/H \to Y/G$$

Note that $N(H)/H$ acts by deck transformations of $Y/H \to Y/G$ (In fact, it should be the full deck group.)

Remark 4.19. When we constructed the cover $X_H \to X$ associated to a subgroup $H \leq \pi_1(X, x)$, we secretly took the quotient of the universal cover $\widetilde{X}$ by the action $H \leq \pi_1(X, x) \cong G(\widetilde{X}) \curvearrowright \widetilde{X}$.

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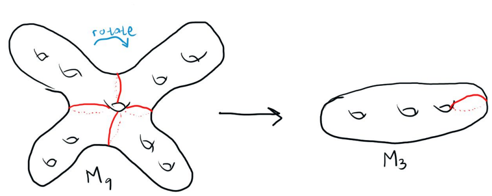
Example 4.20. The surface M_9 has a 4-fold rotational symmetry giving a covering space action of $\mathbb{Z}_4$ with quotient the surface M_3 of genus 3.



Example 4.21. For $X = \mathbb{S}^1$ the circle, the universal cover is $\widetilde{X} = \mathbb{R}$.

We know this we know $\mathbb{R} \to \mathbb{S}^1$ is a simply connected covering space, by the classification it is the universal cover

- The deck group is $G(\mathbb{R}) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$, acting by translations: $k \cdot x = x + k$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$.
- All subgroups of $\pi_1(\mathbb{S}^1)$ have the form $n\mathbb{Z} \leq \mathbb{Z}$ for some $n \in \mathbb{Z}$.
- The cover associated to $H = n\mathbb{Z}$ is the quotient $\mathbb{R}/n\mathbb{Z} = \mathbb{S}^1$ with the covering map $\mathbb{S}^1 \to \mathbb{S}^1$ given by $e^{i\theta} \mapsto e^{in\theta}$.

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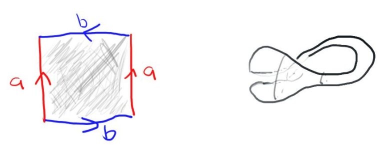
Example 4.22. We've seen that $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ with quotient the torus $\mathbb{T}^2$. Since $\mathbb{R}^2$ is simply connected, this gives another proof that $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$.

Now let G be the group of homeomorphisms of $\mathbb{R}^2$ generated by the translation $(x, y) \mapsto (x + 1, y)$ and the glide reflection $(x, y) \mapsto (-x, y + 1)$. The quotient is the Klein bottle K, formed by identifying opposite sides of a square as shown, which has fundamental group G.

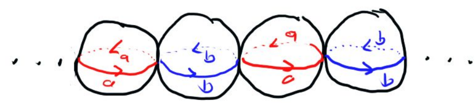


Example 4.23. The antipodal map $x \mapsto -x$ on $\mathbb{S}^n$ generates a covering space action of $\mathbb{Z}_2$ with quotient $\mathbb{R}P^n$. Since $\mathbb{S}^n$ is simply connected for $n \geq 2$, we deduce $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ has order 2 for $n \geq 2$. A generator is any loop γ obtained by projecting a path in $\mathbb{S}^n$ joining two antipodal points. One can see this has order 2, since $\gamma \cdot \gamma$ lifts to a loop in $\mathbb{S}^2$ that is nullhomotopic.

Example 4.24 (Cayley complexes)**.** Recall that for any group presentation $G = \langle g_\alpha \mid r_\beta \rangle$ we built a 2-complex X_G with $\pi_1(X_G) = G$. The universal cover $\widetilde{X}_G$ is called the Cayley complex and is built from the Cayley graph by attaching 2-cells to all loops given by relators.

- For $\mathbb{Z} \times \mathbb{Z} = \langle a, b, \mid aba^{-1}b^{-1} \rangle$, X_G is the torus and the Cayley complex is $\mathbb{R}^2$ with the usual tiling by squares
- For $\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$, X_G is $\mathbb{S}^1 \vee \mathbb{S}^1$ and $\widetilde{X}_G$ is the infinite 4-valent tree.

- For $\mathbb{Z}_2 = \langle a \mid a^2 \rangle$, X_G is $\mathbb{R}P^2$ and $\widetilde{X}_g = \mathbb{S}^2$.
- For $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2, b^2 \rangle$, the Cayley complex $\widetilde{X}_G$ is an infinite chain of spheres



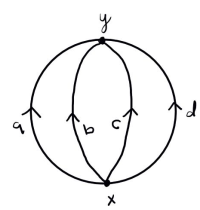
5 *Homology*

Idea of Homology

- $\pi_1(X)$ only depends on low-dimensional information:
  loops $I \to X$ and homotopies of loops
- if X a CW complex, only depends on 2-skeleton!
  - can't distinguish higher dimensional spheres!
- higher dim'l analogs of π_1 (maps $I^n \to X$) difficult to calculate.
- Homology: related to cell structures and rely on higher dim'l info

Example 5.1 (Illustration in low dim)**.**

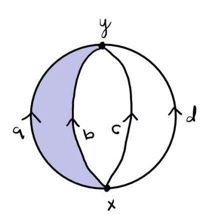
- fund group: pick basepoint and look at loops, eg $ab^{-1}\ldots$



- π_1 nonabelian – abelianize to get $ab^{-1} = b^{-1}a$.
- switch to additive notation $a - b = b - a$
- suggests generalization of taking linear combinations of edges
  - "chains" $ka + \ell b + mc + nd$
  - chain is a "cycle" if every time it enters a vertex it also leaves:
    * here $\iff k + \ell + m + n = 0$

cycles are linear combs where coefficients sum to 0

- Now attach 2–cell with boundary, eg $a - b$.



  - so $a - b$ homotopyically trivial $\rightsquigarrow$ quotient by subgroup generated by $a - b$: Eg $a - c \sim b - c$.
- This generalizes to simplicial homology for Δ–complexes

5.1 *Algebra Review*

Recall, a **chain complex** A_* is seq of abelian groups with $\partial^2 = 0$:

$$\cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \dots$$

The **homology** of A_* is the graded abelian group H_* defined by

$$H_n = \ker(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

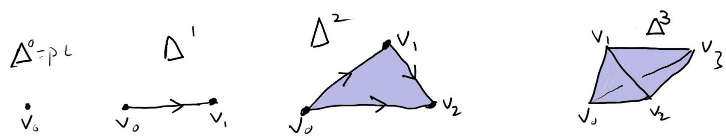
- elts of $\ker(\partial_n)$ called **cycles**
- elts of $\mathrm{im}(\partial_{n+1})$ called **boundaries**
- elts of H_n called **homology classes**. Two cycles representing the same class are **homologous** (means differenence is a boundary)

The homology of a spaces is generally given by building a chain complex and taking its homology.

5.2 *Simplicial homology*

Recall: n-simplex: smallest convex set in $\mathbb{R}^m$ containing $n+1$ points that do not lie in a hyperplane of dimn less than n.

standard n-simplex $\Delta^n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\right\}$



(usually order vertices)

- An n–simplex has $n+1$ **faces** given by deleting a vertex and taking convex hull.
  - (each face an $n-1$ simplex)
- union of all faces = boundary $\partial\Delta^n$
- interior $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$

Definition 5.2. A **Δ–complex structure** on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$ s.t.:

n varies with α

- $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective and each point is in the image of exactly one such restriction
- restriction of σ_α to a face is a $\sigma_\beta \colon \Delta^{n-1} \to X$ (via order-preserving isom.)

- $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

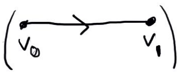
so, we build X by gluing together simplices

Let $\Delta_n(X)$ be the free abelian group on n simplices of X:

$$\Delta_n(X) \text{ consists of finite formal linear sums } \sum k_\alpha \sigma_\alpha$$

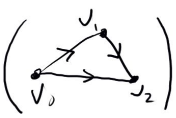
boundary of n–simplex is bunch of $n-1$ simplices $\rightsquigarrow$ **boundary homomorphism**

$$\partial_n \colon \Delta_n(X) \longrightarrow \Delta_{n-1}(X) \qquad \delta_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

$\partial\Big($ 

$\Big) = v_1 - v_0$

signs

$\partial\Big($ 

$\Big) = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$

★ Check $\partial^2 = 0 \rightsquigarrow$ chain complex $\rightsquigarrow$ homology!

Definition 5.3. The **simplicial homology groups** of a Δ–complex X are

$$H_n^\Delta(X) = n^{\text{th}} \text{ homology of chain cplx } \to \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to$$

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